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An experimental investigation of high Reynolds number steady streaming generated by oscillating cylinders

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The steady streaming generated in the boundary layer on a cylinder performing simple harmonic motion in a viscous incompressible fluid which is otherwise at rest is investigated in the case where the Reynolds number R_s associated with this streaming is large. Comparison is made between experimental results obtained here and the theories of Riley (1965) and Stuart (1966). This comparison shows good agreement between the theories and the experiment close to the cylinder, but away from the cylinder significant discrepancies are observed. Possible reasons for these discrepancies are discussed.

1. Introduction

In this paper the steady streaming induced by an oscillating cylinder in a viscous incompressible fluid is investigated experimentally in the case where the Reynolds number R_s associated with this streaming is large. The motion of the fluid is rendered visible by tracer particles, which are photographed using stroboscopic illumination synchronized to the frequency of oscillation. We focus attention upon the streaming in the steady boundary layer. This streaming has been investigated theoretically by Stuart (1963, 1966) and Riley (1965, 1967). Davidson & Riley (1972) investigated this streaming theoretically and experimentally, but paid special attention to the jet emerging along the axis of oscillation. We shall mainly be concerned with the boundary-layer streaming at the cylinder and therefore the theoretical results of Riley (1965) and Stuart (1966) will be those of main interest. We mention that it has been shown by Wiig (1970), who developed Stuart's solution in a power series in ξ (ξ being as defined by Riley 1965), that the results of Riley and Stuart are identical at least to order ξ^3.

From the theory of Riley (1965) we construct the following uniformly valid expression for the time-averaged Eulerian stream function:

$$\psi_R^E(x,y) = U_\infty\{\epsilon(2\nu/\omega)^{\frac{1}{2}}[-4\sin(4\xi)\chi_{20}(\eta)+\tfrac{3}{4}S(\xi)\eta] + (\nu/\omega)^{\frac{1}{2}}[c_0\xi\phi_0(\zeta)+c_1\xi^3\phi_1(\zeta)+c_2\xi^5\phi_2(\zeta)]\}, \quad (1)$$

where
$$S(\xi) = -(c_0^2\xi+c_0c_1\xi^3+c_0c_2\xi^5), \quad (2)$$

with $c_0 = 4$, $c_1 = -\frac{32}{3}$ and $c_2 = \frac{128}{15}$. The functions $\chi_{20}(\eta)$, $\phi_0(\zeta)$, $\phi_1(\zeta)$ and $\phi_2(\zeta)$ are given by Riley (1965), see his equations (8), p. 163, and (27)–(29), p. 167, respectively. Further, U_∞ is the velocity amplitude of the oscillating fluid, $\epsilon = U_\infty/2a\omega$,

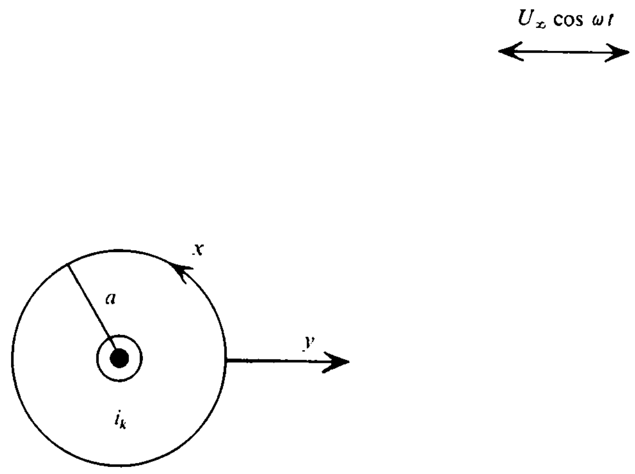


FIGURE 1. The (x, y) co-ordinate system referred to in the text.

a the radius of the cylinder, ω the angular frequency of the oscillations and ν the kinematic viscosity. The non-dimensional variables ξ, η and ζ are defined as $\xi = (x-x_0)/2a$ with $x_0/2a = \frac{1}{4}\pi$, $\eta = (\omega/2\nu)^{\frac{1}{2}}y$ and $\zeta = 4\epsilon(\omega/\nu)^{\frac{1}{2}}y$. The (x, y) co-ordinate system is the physical co-ordinate system indicated in figure 1 (x being measured along and y orthogonal to the cylinder surface). In the theory the fluid is supposed to oscillate as $U_\infty \cos \omega t$ infinitely far from the cylinder. In the experiment the cylinder oscillates, but this does not affect the time-averaged Lagrangian velocity observed (see Bertelsen, Svardal & Tjøtta 1973).

From the theory of Stuart (1966) we construct the following uniformly valid expression for the time-averaged Eulerian stream function:

$$\psi_S^E(x,y) = U_\infty \left\{ \epsilon \left(\frac{2\nu}{\omega}\right)^{\frac{1}{2}} V \frac{dV}{d\xi_0} [\chi_{20}(\eta) + \tfrac{3}{4}\eta] + \left(\frac{\nu}{\omega}\right)^{\frac{1}{2}} \phi(\xi_0, \zeta) \right\}, \tag{3}$$

where $\xi_0 = x/2a$, $V(\xi_0) = 2\sin(2\xi_0)$, $\chi_{20}(\eta)$ is as in (1) and $\phi(\xi_0, \zeta)$ is given by equation (3.2) on p. 680 in Stuart's paper. (Notice that Stuart defines $\zeta = \epsilon(\omega/\nu)^{\frac{1}{2}}y$.)

Now we introduce the Stokes drift as a correction to the stream function. This correction is (see Raney, Corelli & Westervelt 1954)

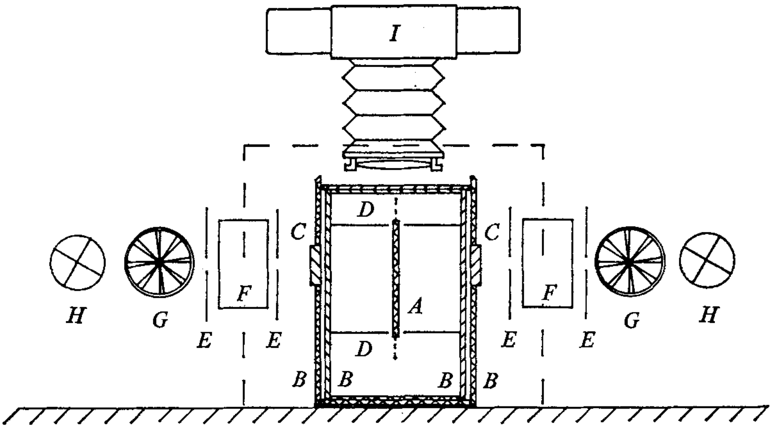
$$\Delta\psi(x,y) = \mathbf{i}_k \cdot \tfrac{1}{2}\langle(\textstyle\int \mathbf{v}_1 dt \times \mathbf{v}_1)\rangle, \tag{4}$$

where $\mathbf{v}_1$ is the first-order unsteady velocity field, $\mathbf{i}_k$ the unit vector orthogonal to the plane of motion and angular brackets indicate a time average. Using the first-order unsteady velocity field derived from the corresponding stream function (for the non-dimensional stream function see, for example, Riley 1965, equation (7), p. 163) equation (4) gives, in physical variables,

$$\Delta\psi(x,y) = 2^{\frac{1}{2}} \frac{R_s}{M} \nu \{2\eta e^{-\eta} \sin\eta + 2e^{-\eta}\cos\eta - e^{-2\eta} - 1\} \sin\frac{2x}{a}, \tag{5}$$

where $M = 2a(\omega/\nu)^{\frac{1}{2}}$ and $R_s = \epsilon^2 M^2$. Thus we have from Riley's theory the time-averaged Lagrangian stream function,

$$\psi_R^L(x,y) = \psi_R^E(x,y) + \Delta\psi(x,y), \tag{6}$$



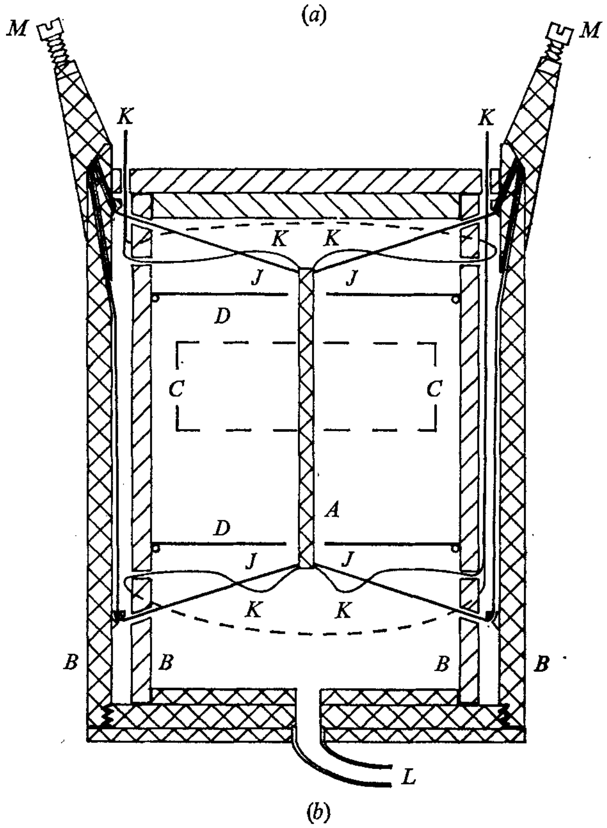
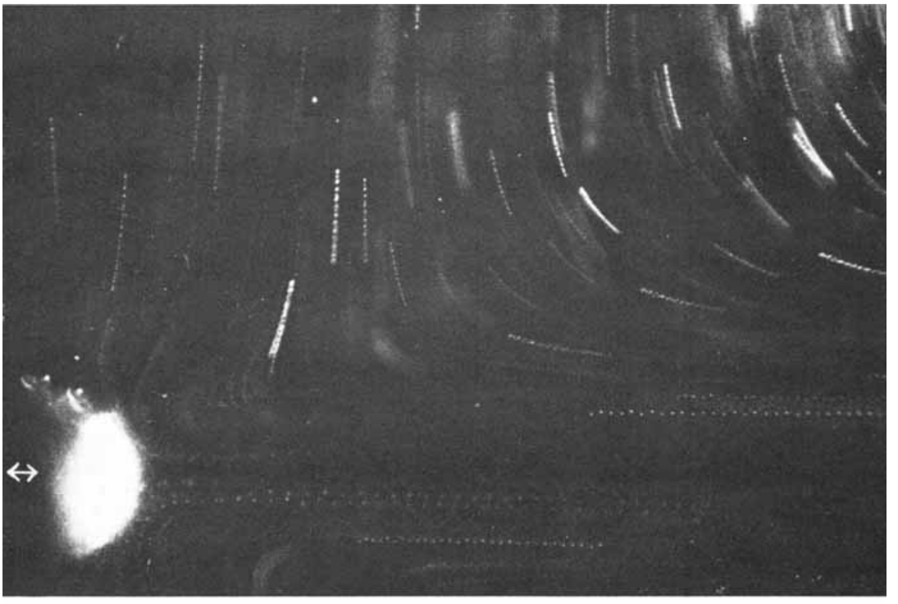
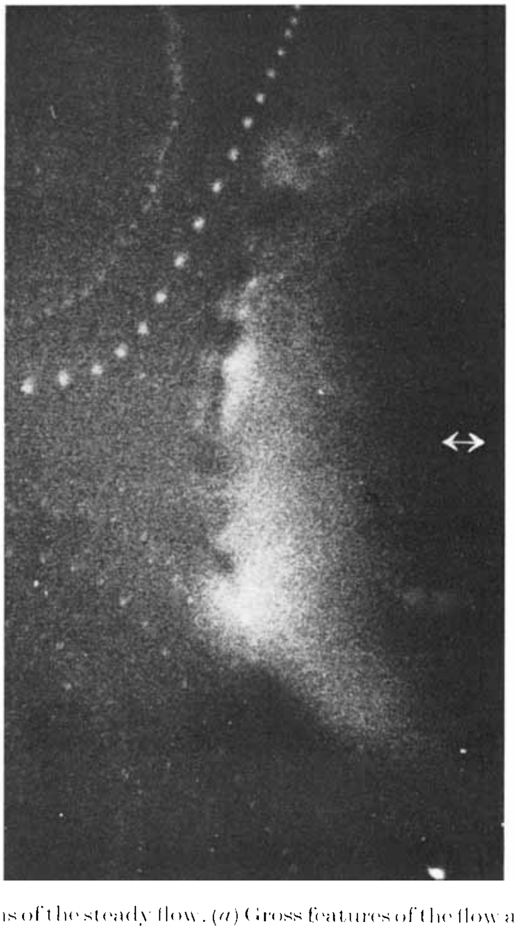


FIGURE 2. Sketch of the apparatus used in the experimental investigation. (*a*) The experimental set-up. *A*, inner oscillating cylinder; *B*, outer boundary with double walls; *C*, windows; *D*, thin circular perspex plates; *E*, slits; *F*, water filter to absorb heat radiation; *G*, fan to cool the stroboscope lamp; *H*, stroboscope lamp; *I*, camera; --- extent of the pole of the magnet. (*b*) The cylindrical box. *A*–*D*, as in (*a*), *J*, suspension lines; *L*, fluid inlet; *M*, turnbuckles for the suspension lines.



(*a*)



(*b*)

FIGURE 4. Photographs of the steady flow. (*a*) Gross features of the flow at $R_s = 90$. (*b*) Details of the flow in the boundary layer at $R_s = 90$. (White arrows indicate direction of oscillation.)

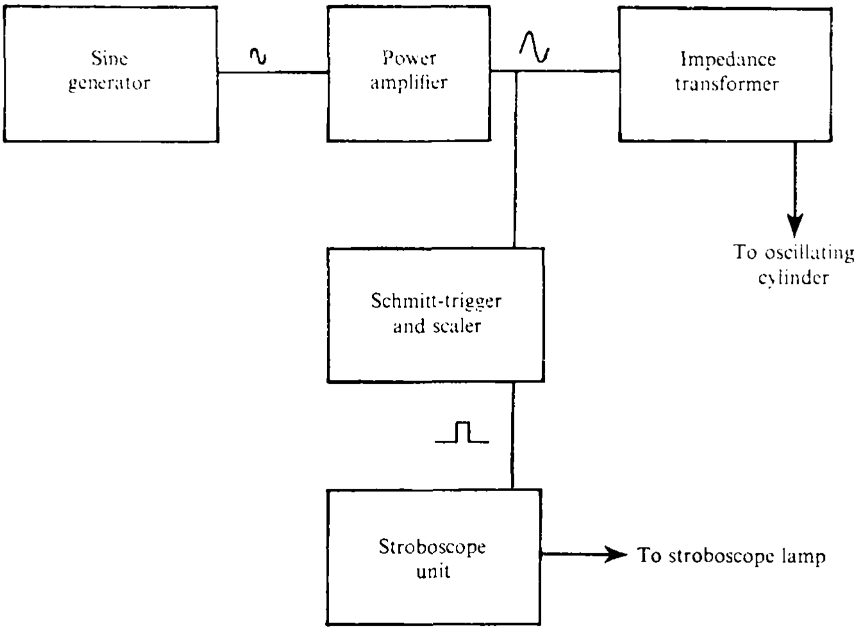


FIGURE 3. Block diagram of the electronics.

and from Stuart's theory,

$$\psi_S^L(x,y) = \psi_S^E(x,y) + \Delta\psi(x,y). \tag{7}$$

The time-averaged Lagrangian tangential velocity from (6) is

$$U_R(x,y) = \partial\psi_R^L/\partial y, \tag{8}$$

and from (7),

$$U_S(x,y) = \partial\psi_S^L/\partial y. \tag{9}$$

The contribution from the Stokes drift to the velocity components given by (8) and (9) is

$$\Delta U = \partial\Delta\psi/\partial y.$$

We note that†

$$\Delta U \to 0 \quad \text{as} \quad \eta \to \infty.$$

Thus the Stokes drift becomes insignificant in our problem as we pass from the innermost region (the shear-wave layer) to the inner region (the steady boundary layer). Numerical values obtained from (8) and (9) are plotted in figures 5 and 6.

2. Apparatus and method of observation

A brief description of the apparatus was given by Bertelsen *et al.* (1973). Further details are added here. A sketch of the apparatus is shown in figure 2. The inner cylinder was a 6·2 cm long brass tube. Different diameters were used. The tube

† In this connexion we should like to point out a printing error in the equation following equation (27) on p. 507 in the paper of Bertelsen *et al.* (1973). The equation should read

$$\lim_{\eta\to\infty} \Delta\psi_i = \lim_{r\to 1} \Delta\psi_y = -\frac{\epsilon}{M\sqrt{2}}\sin 2\theta.$$

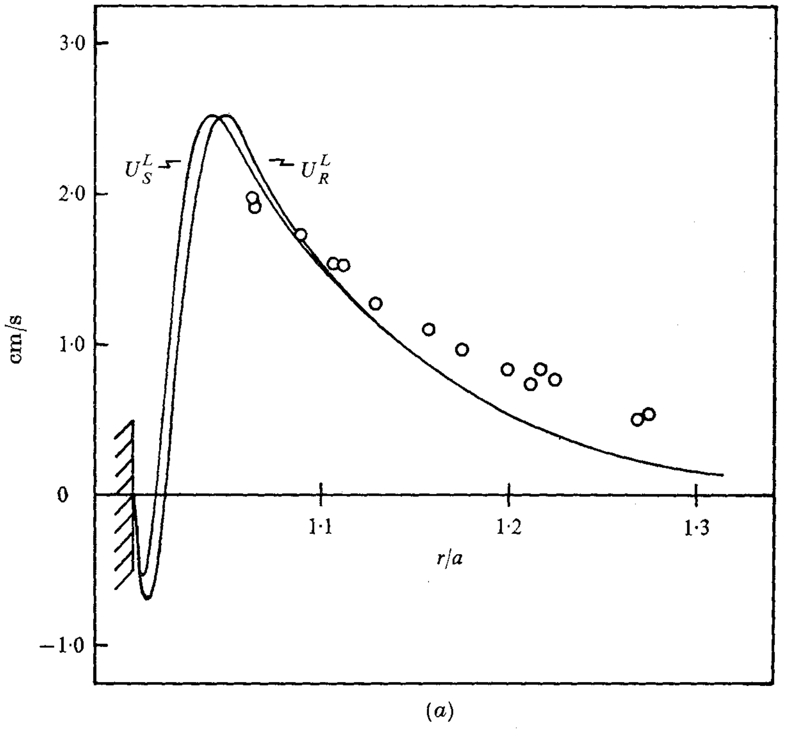
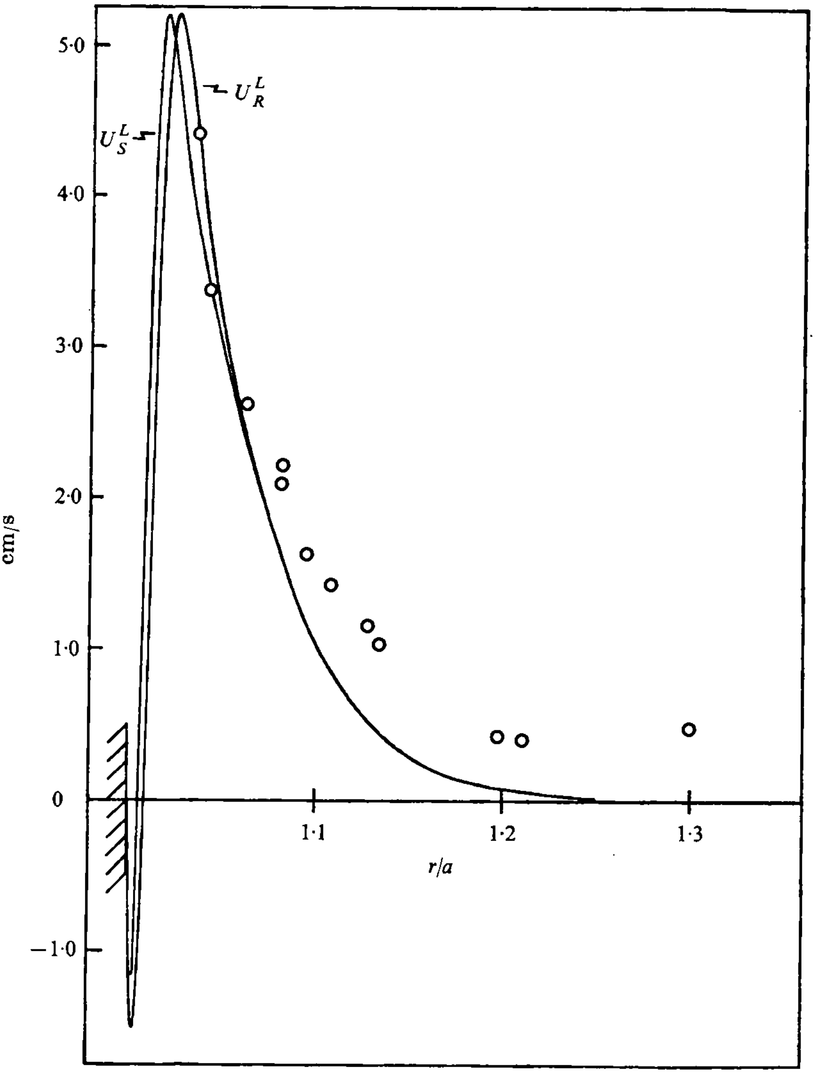


FIGURE 5(*a*). For legend see following page.

was blackened with a thin layer of silver sulphide to avoid reflexion of light. This was necessary in order to be able to study the flow close to the cylinder. The cylinder was suspended with perlon lines, but these are scarcely the best suspension lines as it was several days before the mechanical system achieved a constant resonance frequency. The cylinder was forced to oscillate sinusoidally by being placed in a constant magnetic field ($0{\cdot}15\ \mathrm{Wb/m^2}$) and supplied with an alternating current $\sim 0{\cdot}5$ A. Sufficiently large amplitudes were obtained only when the driving force had a frequency close to the resonance frequency of the mechanical system.

The outer boundary was a cylindrical tube with double walls to minimize convection streaming caused by temperature gradients. It was also necessary to cool the stroboscope lamp to avoid such streaming. A bottom of brass and cover of perspex closed the cylindrical box. The box was filled with fluid (pentane or water) with tracer particles (MnO powder) added. The tracer particles were photographed using stroboscopic illumination synchronized to the frequency of oscillation or an integral part of this. To get good contrast in the photographs it was preferable to illuminate only a thin layer of the fluid. The tracer particles appear as white dots on the photographs (see figure 4, plate 1). From such photographs the time-averaged Lagrangian velocity is readily obtained as the repetition frequency of the stroboscope is known and the distance between dots on the same streamline can be measured. Some electronic equipment was also necessary and a block diagram of this is shown in figure 3.



(b)

FIGURE 5. Time-averaged Lagrangian tangential velocity at $\xi = 0{\cdot}5$. (a) $R_s = 90$; other parameters given in table 1, series I. (b) $R_s = 400$; other parameters given in table 1, series II. U_S^L, from Stuart's theory; U_R^L, from Riley's theory; ○, measured values.

3. Results and discussion

Two series of measurements of the steady tangential velocity in the boundary layer were performed at $\xi \approx 0{\cdot}5$ and the parameters were as given in table 1, series I and II. The main features of the streaming were as shown in figure 4 (plate 1). Figures 5(a) and (b) show the measured velocities for series I and II, respectively, compared with the corresponding theoretical velocity profiles. This comparison shows that the agreement between the theories of Riley (1965) and Stuart (1966) and our experiment is good close to the inner cylinder, but at distances larger than δ_s ($= 2aR_s^{-\frac{1}{2}}$) there are significant discrepancies.

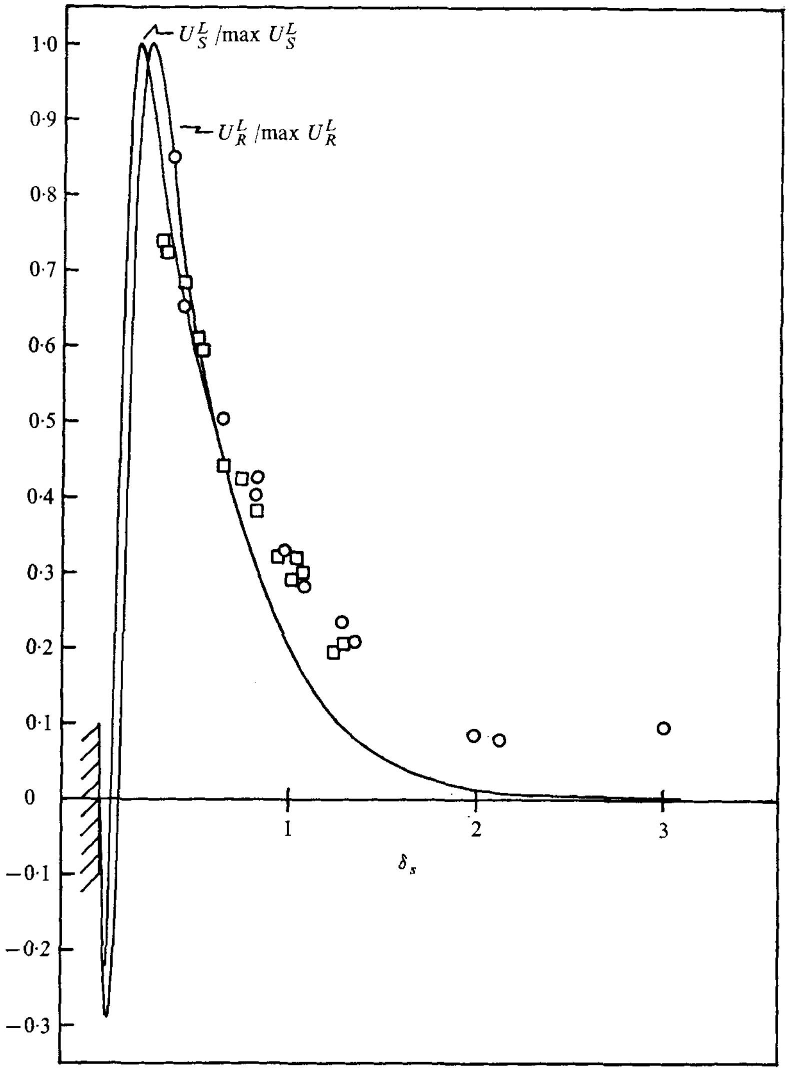


FIGURE 6. Normalized plot of figures 5(*a*) and (*b*). Reference velocity is maximum of theoretical steady velocity in the boundary layer and $\delta_s = 2a(R_s)^{-\frac{1}{2}}$, the steady boundary-layer thickness. Squares are measured values: □, $R_s = 90$; ○, $R_s = 400$.

Series	Amplitude $\epsilon = U_\infty/2\omega a$	Ratio $2a/\delta_{ac}$ $M = 2a(\omega/\nu)^{\frac{1}{2}}$	Reynolds number $R_s = \epsilon^2 M^2$	Radius of oscillating cylinder a (cm)	Radius of outer boundary A (cm)	Frequency of oscillation f (Hz)	Kinematic viscosity ν (cm²/s)	Velocity amplitude U_∞ (cm/s)
I	$\frac{1}{24}$	227	90	0·150	3·10	350	0·00383	27·5
II	$\frac{1}{20}$	400	400	0·307	4·00	260	0·00383	50·1
III	$\frac{1}{24}$	187	60	0·307	4·00	150	0·0100	24·0

TABLE 1

In figure 6 a normalized plot of figures 5 (*a*) and (*b*) is given and the maximum theoretical steady velocity in the boundary layer is used as a reference velocity. Here we remark that the measurements of series I and II coincide within expected limits. From this we conclude that the measurements are reliable and we have an indication that a typical steady boundary layer develops when $R_s \sim 100$.

There are several possible reasons for the discrepancies mentioned above. In the experiment there is an outer boundary of radius A which is not accounted for in the theories, but this is not very important since (see Stuart 1966, equation (5.2)) $(\delta_s/A)^2 \sim 10^{-4}$ in our experiment. Further, the theories do not describe curvature effects, displacement streaming nor the effect on the boundary-layer streaming of vorticity in the flow outside the boundary layer. Such effects are of order $R_s^{-\frac{1}{2}}$ (see Van Dyke 1962) relative to the velocity given by the theories. In the experiment of series I, $R_s^{-\frac{1}{2}} \sim 0{\cdot}1$ and in series II, $R_s^{-\frac{1}{2}} \sim 0{\cdot}05$, which indicates that these effects may be important in explaining the discrepancies mentioned above. Another important feature of the steady streaming is the separation points at $\xi = \pm\frac{1}{4}\pi$, where the impinging boundary layers separate from the cylinder in a jet-like flow. This separation can alter the flow outside the boundary layer some distance upstream of the point of separation and thus affect the boundary-layer streaming itself (see Van Dyke 1964). In this connexion it is an important feature of the experimental results that increasing the Reynolds number R_s from 90 to 400 does not improve the agreement between theory and experiment at the outer edge of the steady boundary layer.

Other properties of the flow were also studied. Thus we found $M \approx 200$ to be the largest value of M for which it was possible to observe the inner vortex systems in our experiment. For the parameters given in table 1, series III we determined the thickness δ_{DC} of the inner vortex systems to be within

$$0{\cdot}003\,\text{cm} < \delta_{DC} < 0{\cdot}009\,\text{cm}.$$

This thickness is defined as the distance between the cylinder and the first zero-point of the steady radial velocity component. Our experimental value is consistent with the theory, which predicts

$$\delta_{DC} \approx 0{\cdot}007\,\text{cm}.$$

The jet emerging from the boundary layer in the direction of oscillation was observed (see Davidson & Riley 1972). We found that the jet became unstable for $R_s > 100$. The instability first appeared at the stagnation point where the outgoing jet met the outer boundary and for $R_s = 400$ the jet was unstable as close to the inner cylinder as $y/a = 1{\cdot}0$.

The uncertainty in the measurements of the velocity field is approximately 10 %. This uncertainty is mainly due to variations in the amplitude of oscillation.

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